Coupling bosonic modes with a qubit: entanglement dynamics at zero and finite-temperatures

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(Dated: February 1, 2008)

We consider a system of two iso-spectral bosonic modes coupled with a single two-level systems i.e., a qubit. The dynamics is described by a mode-symmetric two-modes Jaynes-Cummings. The entanglement, induced between the two bosonic modes, is analyzed and quantified by negativity. We computed the time evolution of negativity starting from an initial thermal state of the bosonic sector for both zero and finite temperature. We also studied the entangling power of the interaction as a function of mode-qubit detuning and its resilience against temperature increase. Finally a two-qubit gates based on bosonic virtual subsystem is discussed.

INTRODUCTION

In several of the existing implementation proposals for quantum information processing [1] hybrid system are involved. By hybrid here we simply mean that both discrete i.e., atomic, and continuous e.g., vibrational modes in an ion-trap [2], degrees of freedom are present and interacting. Typically the former are chosen to play the role of quantum information carrier whereas the latter are treated either as information buses (to couple on demand remote discrete systems e.g., cavity modes coupling atoms [3, 4]) or are hold responsible for decoherence effects e.g., phonons in quantum dots. In either cases the attention is, in a sense, mostly focused on the information-carrying discrete sector, and the bosonic modes are eventually traced away.

In this paper we would like, so to speak, to reverse this logic and to address the problem of bosonic modes interacting using discrete systems as interaction medium. We will consider the simplest instance of this situation: two iso-spectral bosonic modes coupled through the interplay of one qubit. We consider here the bosonic degrees of freedom as the relevant ones for encoding information, while the qubit degrees of freedom will be traced out. Information will be encoded in a finite-dimensional subspace of the two-mode bosonic Fock space.

Of course the use of continuous variables i.e., light modes in the context of quantum information processing has been already massively investigated in the recent literature. On one hand there is the effort to handle with the infinite dimensional Fock space of bosonic degrees of freedom by mapping it onto finite dimensional ones by choosing ad hoc states. On the other hand a careful choice of a suitable subspace of the Fock space can also provide a proper way for encoding and processing qubits [5] The first approach is the one contained e.g. in [6] in which the gaussian states are involved in view of their key feature of being easily mappable onto the finite dimensional space covariance matrices. The second approach instead aims at finding information-encoding states whose time evolutions remains inside a finite dimensional subspace, at least for properly selected time instants [7, 8, 9].

This second approach is the one followed in this paper. Given the interaction Hamiltonian, a proper encoding i.e., a two-dimensional Fock subspace, is searched. The require-ment that one can individuate time instants in which a non-trivial unitary transformation of the encoding subspace is enacted.

More specifically, in this paper we shall analyze a symmetric two-modes Jaynes-Cummings model. The entanglement properties of the bosonic subsystem are investigated by computing the dynamical evolution of the negativity [10] of a generic state of the system. Analytical and numerical results respectively are found for the vacuum and the thermal initial state. The latter case is also studied to prove the entanglement resilience against temperature and qubit-modes detuning. The entanglement capabilities of our system are then analyzed by introducing a convenient notion of entangling power [11, 12] and by observing its behaviour as a function of the model parameters. Finally a unitary quantum gate for this kind of interaction, is contrived. The realization of this gate however will require the use of virtual subsystems [13] and of the related entanglement relativity concept [14].

THE MODEL

The physical system studied in this paper is the one described by the two-mode Jaynes-Cummings hamiltonian [7, 8]:

$$H = \sigma_z \epsilon + \sum_{i=1}^{2} \hbar\omega_i b_i^\dagger b_i + \sum_{i=1}^{2} \gamma(\sigma_- b_i^\dagger + \sigma_+ b_i) \quad (1)$$

The symbols b_i and $b_i^\dagger$ stand for annihilation and creation of a particle (photon or phonon) in the mode i, while γ is the coupling constant and ϵ is the energy difference between the two states. The hamiltonian simplifies when i) the two modes have the same energy $\omega_1 = \omega_2 = \omega$, and ii) the system achieves resonance $\epsilon = \hbar\omega$. For the sake of simplicity we deal first with this case. In order to diagonalize the hamiltonian we perform a unitary transformation of the two mode operators, introducing two new modes: $b_+ = \frac{1}{\sqrt{2}}(b_1 + b_2)$ and $b_- = \frac{1}{\sqrt{2}}(b_1 - b_2)$, as well as their conjugated $b_+^\dagger, b_-^\dagger$, and the inverse equations: $b_1 = \frac{1}{\sqrt{2}}(b_+ + b_-)$ and $b_2 = \frac{1}{\sqrt{2}}(b_+ - b_-)$. The

hamiltonian then becomes:

$$H = \sigma_z \epsilon + \hbar\omega b_+^\dagger b_+ + \gamma(\sigma_- b_+^\dagger + \sigma_+ b_+) + \hbar\omega b_-^\dagger b_- \quad (2)$$

As we can see, now the hamiltonian is the single-mode Jaynes-Cummings hamiltonian plus a commuting term (the last one). The interaction involves now only one mode while the other does not evolve. The evolution of a generic (pure) state is now easy to obtain following for example ref.[17, 18]; if the system is prepared in the initial state with the qubit in its excited state (e) and the two bosonic modes occupied respectively with n_1 and n_2 particles ($|\psi\rangle = |e, n_1, n_2\rangle = \sum_{n_+,n_-} a_{n_+,n_-}|e, n_+, n_-\rangle$), the evolution will be:

$$
\begin{aligned}
|\psi(t)\rangle &= \sum_{n_1,n_2} [A_{n_1,n_2}(t)|e, n_1, n_2\rangle \\
&+ B_{n_1,n_2}(t)|g, n_1+1, n_2\rangle + C_{n_1,n_2}(t)|g, n_1, n_2+1\rangle] \\
&= \sum_{n_+,n_-} a_{n_+,n_-}[c_{n_+}(t)|e, n_+, n_-\rangle \\
&- is_{n_+}(t)|g, n_++1, n_-\rangle]
\end{aligned}
\quad (3)
$$

where $c_n(t) = \cos\Omega_n t$, $s_n(t) = \sin\Omega_n t$ and $\Omega_n = \gamma\sqrt{n+1}$ is the (half) Rabi frequency. The coefficients a_{n_+,n_-} are obtained by writing the initial state as a sum of states with fixed number of particles in the new modes:

$$
\begin{aligned}
|\psi\rangle &= |e, n_1, n_2\rangle = \frac{1}{\sqrt{n_1! n_2!}} b_1^{\dagger n_1} b_2^{\dagger n_2} |e, 0, 0\rangle \\
&= \left(\frac{1}{\sqrt{2}}\right)^{n_1+n_2} \frac{1}{\sqrt{n_1! n_2!}} (b_+^\dagger + b_-^\dagger)^{n_1} (b_+^\dagger - b_-^\dagger)^{n_2} |e, 0, 0\rangle \\
&= \left(\frac{1}{\sqrt{2}}\right)^{n_1+n_2} \frac{1}{\sqrt{n_1! n_2!}} \times \\
&\times \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i}\binom{n_2}{j}(-1)^j b_+^{\dagger\, i+j} b_-^{\dagger\, n_1+n_2-i-j}|e, 0, 0\rangle \\
&= \sum_{n_+,n_-} a_{n_+,n_-}|e, n_+, n_-\rangle
\end{aligned}
\quad (4)
$$

having identified, in the last passage, $i+j$ and n_1+n_2-i-j with n_+ and n_- respectively.

With this formalism we can in principle describe the evolution of a generic density operator:

$$
\begin{aligned}
\rho &= \sum_{n_1,n_2} p_{n_1,n_2} \rho_{n_1,n_2} \\
&= \sum_{n_1,n_2} p_{n_1,n_2} |e, n_1, n_2\rangle\langle e, n_1, n_2| = \sum_{n_1,n_2} p_{n_1,n_2} \times \\
&\times \left[\sum_{n_+,n_-} \sum_{n_+',n_-'} a_{n_+,n_-} a_{n_+',n_-'}' |e, n_+, n_-\rangle\langle e, n_+', n_-'| \right]
\end{aligned}
\quad (5)
$$

In general this straightforward procedure can be quite difficult to handle because of the big amount of terms involved in the sum. However in two cases the expression of the time evolution of the initial state highly simplifies. They are the vacuum and the thermal state cases.

ENTANGLEMENT DYNAMICS

In this section we will study the bosonic entanglement dynamics enacted by (1); we will start from different initial preparations i.e., zero and finite temperature. As entanglement measure we adopt *negativity* introduced in Ref. [10]

Vacuum State

If the system is initially in the ground state with respect to the two bosonic modes, it is straightforward to obtain: $\rho_{n_1,n_2} = \rho_{n_+,n_-} = |e, 0, 0\rangle\langle e, 0, 0|$ being $n_i = n_\pm = 0$. The time evolution of this state is given by:

$$
\begin{aligned}
\rho(t) &= c_0^2(t)|e, 0, 0\rangle_\pm\langle e, 0, 0| + s_0^2(t)|g, 1, 0\rangle_\pm\langle g, 1, 0| \\
&+ ic_0(t)s_0(t)(|e, 0, 0\rangle_\pm\langle g, 1, 0| + |g, 1, 0\rangle_\pm\langle e, 0, 0|)
\end{aligned}
\quad (6)
$$

Here and in the following we denote with $|\rangle_{12}$ and $|\rangle_\pm$ the elements of the two basis sets $\{|n_1\rangle \otimes |n_2\rangle\}$ and $\{|n_+\rangle \otimes |n_-\rangle\}$ respectively.

We choose to use the negativity [10] as a measure of entanglement for bipartite states. The negativity is defined as the sum of the negative eigenvalues of the partial transpose (transposed with respect of one of the two subsystems) [19, 20] of the density matrix of the state. In order to compute the negativity of the state with respect to the bipartition defined by the two bosonic modes we trace out the qubit degrees of freedom:

$$
\begin{aligned}
\rho(t) &= c_0^2(t)|0, 0\rangle_\pm\langle 0, 0| + s_0^2(t)b_+^\dagger|0, 0\rangle_\pm\langle 0, 0|b_+ \\
&+ c_0^2(t)|0, 0\rangle_{12}\langle 0, 0| + \frac{s_0^2(t)}{2}(b_1^\dagger + b_2^\dagger)|0, 0\rangle_{12}\langle 0, 0|(b_1 + b_2) \\
&= c_0^2(t)|0, 0\rangle_{12}\langle 0, 0| + \frac{s_0^2(t)}{2}(|1, 0\rangle_{12}\langle 1, 0| \\
&+ |1, 0\rangle_{12}\langle 0, 1| + |0, 1\rangle_{12}\langle 1, 0| + |0, 1\rangle_{12}\langle 0, 1|)
\end{aligned}
\quad (7)
$$

The reduced density operator can be now represented as a matrix in the $|n_1\rangle \otimes |n_2\rangle$ basis, and its partial transpose ρ^{T_1} with respect to (e.g.) the subsystem of mode 1 reads:

$$
\rho^{T_1} = \begin{pmatrix} c_0^2 & 0 & 0 & \frac{1}{2}s_0^2 \\ 0 & \frac{1}{2}s_0^2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}s_0^2 & 0 \\ \frac{1}{2}s_0^2 & 0 & 0 & 0 \end{pmatrix}
\quad (8)
$$

The negative part of the spectrum of this matrix gives the analytical result for the negativity:

$$\mathcal{N} = |\frac{1}{2}c_0^2 - \frac{1}{2}\sqrt{c_0^4 + s_0^4}| \quad (9)$$

whose graphical plot is displayed in fig.(1). It is interesting to note the periodic behaviour of the negativity which vanishes with period π/γ. On the other hand we can see that this kind of interaction is able to entangle very much the system, giving raise to maximally entangled states with the same period π/γ.

In fact as $t = \pi/2\gamma + K(\pi/\gamma)$ the state vector is (except for a phase factor):

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle_{12} + |0,1\rangle_{12}) \qquad (10)$$

In fig.(1), besides the negativity, we plotted the linearized Von Neumann entropy too: $S_L(\rho) = 1 - Tr_1(Tr_2\rho)^2$, with Tr_i meaning trace over the subsystem i. Its analytical expressions for the present state is given by:

$$S_L = 1 - c_0^4 - \frac{1}{2}s_0^4 \qquad (11)$$

As long as $\rho(t)$ is a pure state, the Von Neumann entropy is a measure of entanglement and its maximum coincides with the negativity of the maximally entangled state. The state $\rho(t)$ is pure when $t = \pi/2\gamma + K(\pi/2\gamma)$, as it is easy to recognize by computing its spectrum: $\sigma[\rho(t)] = [0, 0, c_0^2(t), s_0^2(t)]$. It is worth noting that when the state is no more pure the Von Neumann entropy may exceed the maximum entanglement value for pure states, then being a no more reliable measure of entanglement, as fig.(1) shows.
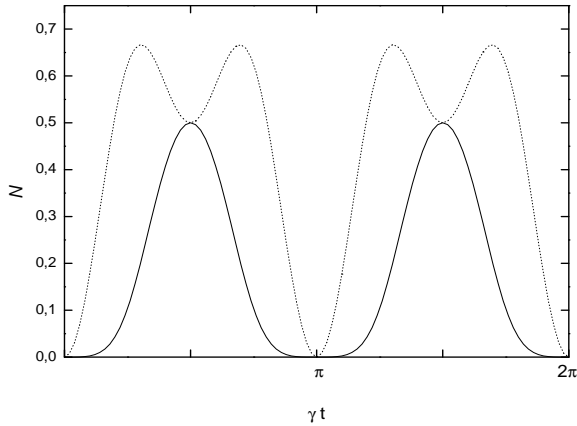

FIG. 1: Negativity as a function of time for the vacuum state (solid line). The linearized Von Neumann entropy is plotted in dots for comparison: the two lines touch each other when ρ is a pure state.

This result holds even for the more general case of a multi-mode hamiltonian, and provides a quick way to generate the so-called W n-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{n}}(|1, ..., 0\rangle + ... + |0, ..., 1\rangle) \qquad (12)$$

This feature will be exploited in the design of a quantum gate in the following.

Thermal State

Let us move now to the case in which both bosonic modes are in a thermal state. The bosonic system will be described at the initial time as a tensor product of the two thermal states:

$$\rho = e^{-\beta\hbar\omega n_1} \otimes e^{-\beta\hbar\omega n_2} = e^{-\beta\hbar\omega(n_1+n_2)} = e^{-\beta\hbar\omega(n_++n_-)} \qquad (13)$$

where, as usual, $\beta = \frac{1}{k_B T}$ and n is used in the operatorial meaning ($n = b^\dagger b$). The last identity of eq. (13) allows us to write the density operator directly in the $|n_+\rangle \otimes |n_-\rangle$ basis: $\rho = \sum_{n_1,n_2} p_{n_1,n_2}|n_1, n_2\rangle\langle n_1, n_2| = \sum_{n_+,n_-} p_{n_+,n_-}|n_+, n_-\rangle\langle n_+, n_-|$ where the sum goes as usual from zero to infinity. The coefficients p_n account for the Bose-Einstein probability distribution, according to which:

$$p_n = \frac{\langle n\rangle^n}{(1 + \langle n\rangle)^{n+1}} \qquad (14)$$

and the mean boson number is

$$\langle n\rangle = \frac{1}{e^{\beta\hbar\omega} - 1} \qquad (15)$$

The time evolution of this thermal state, after tracing out the qubit, will be:

$$\rho(t) = \sum_{n_+,n_-} p_{n_+,n_-}[c_{n_+}^2(t)|n_+, n_-\rangle\langle n_+, n_-| + \\ + s_{n_+}^2(t)|n_+ + 1, n_-\rangle\langle n_+ + 1, n_-|] \qquad (16)$$

The matrix form of this operator, of course, is an infinite square matrix whose spectrum can be computed numerically thanks to the sparse structure of the matrix itself. By increasing the temperature the higher energy levels begin to contribute to the density and the infinite sum of terms has to be truncated carefully, taking into account more and more levels. The matrix elements of ρ^{T_1} can be written as a sum of matrices with fixed number of particles weighted with their thermal probability distribution. By combining eq.(4) and (5) as well as their inverse, it is possible, after some tedious but straightforward calculations, to write down the matrix elements of ρ^{T_1} as:

$$\rho_{\mu\nu}^{T_1} = \lim_{N\to\infty} \sum_{n=0}^{N} [\sum_{d_1=0}^{n} c_{d_1}^2(t) \frac{1}{2^n} \sqrt{\binom{d_1}{i}\binom{d_1}{j}} \sqrt{\binom{d_2}{k}\binom{d_2}{l}} (-1)^{k+l} p_{d_1} p_{d_2} \delta_{\mu,n+1+(N+2)(i+j)-(k+l)} \delta_{\nu,n+1+(N+2)(k+l)-(i+j)} +$$

$$+ \sum_{d_1'=1}^{n+1} s_{d_1'-1}^2(t) \frac{1}{2^n} \sqrt{\binom{d_1'}{i}\binom{d_1'}{j}} \sqrt{\binom{d_2'}{k}\binom{d_2'}{l}} (-1)^{k+l} p_{d_1'-1} p_{d_2'} \delta_{\mu,n+2+(N+2)(i+j)-(k+l)} \delta_{\nu,n+2+(N+2)(k+l)-(i+j)}] \qquad (17)$$

where $d_2 = n - d_1$; $d_2' = n + 1 - d_1'$ and $i, j = 0, ..., d_1$; $k, l = 0, ..., d_2$.

The matrix has a multi-diagonal form, meaning that the non-zero elements are placed in diagonal lines parallel to principal diagonal of the matrix. The computation of its spectrum and in particular of its negative part (negativity) has been done in a numerical way. The series appearing in eq.(17) has been truncated using as a convergence criterion: $\sum_{n=0}^{N} \sum_{d_1=0}^{n} p_{d_1} p_{d_2} \sim 1$. The results are shown in fig.(2) in which the time evolution of the negativity is plotted for different values of the energy/temperature ratio $\eta = \frac{\hbar\omega}{k_B T}$. As one can expect, the negativity peaks get lower as temperature raises. Nevertheless, surprisingly enough, the negativity remains different from zero for long time even at high temperature. At $T = 0$ we recover the same results of the previous subsection.
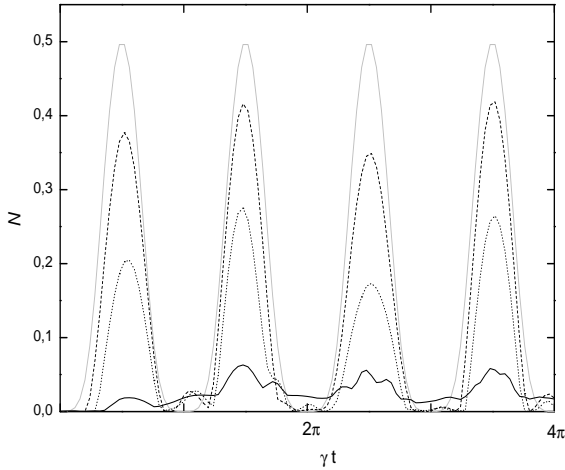

FIG. 2: The negativity behaviour of the thermal state as a function of time is plotted for four different temperature. The latter are labelled with different values of the parameter $1/\eta$: 0 for the solid thin gray line, 0.5 for the dashed line, 1 for the dotted line, 5 for the solid black line.

In order to make our analysis slightly more general, we now extend these results to the case in which the qubit and the bosons are off-resonance. This occurs when the qubit energy splitting and the bosonic mode frequencies are different. In this situation the coefficients c_n and s_n have a more complicated time dependence [17]:

$c_n = [\cos(\Omega_n t) - \frac{i\Delta}{2\Omega_n}\sin(\Omega_n t)]\exp(i\Delta t/2)$ and $s_n = (i\gamma\sqrt{n+1}/\Omega_n)\sin(\Omega_n t)\exp(i\Delta t/2)$, while the (half) Rabi frequency becomes: $\Omega_n = \frac{1}{2}\sqrt{\Delta^2 + 4\gamma^2(n+1)}$, with $\Delta = (\epsilon - \hbar\omega)/\hbar$. Fig.(3) shows the negativity behaviour at $T = 0$ for increasing values of Δ, while fig.(4) shows how the negativity varies in time for different temperatures with a fixed non-vanishing value of the detuning ($\Delta = 1$). As a general remark, it is worth noting that an increasing detuning causes the peaks of the negativity to lower at fixed temperature. In a similar way the high temperatures lower the negativity as already known from the resonance case (fig.(2) and (4) are quite similar). Finally for high values of Δ the negativity oscillates with frequency γ^2/Δ as a result of a second order effective interaction hamiltonian H_{eff}:

$$H_{eff} = \hbar\Omega_{eff} b_1^\dagger b_2 + h.c. \qquad (18)$$

where $\Omega_{eff} = \gamma^2/\Delta$. In fig.(3) the dotted line plots the negativity for $\Delta = 2$ and $\gamma = 1$. Since $\Delta >> \gamma$ we can consider H_{eff} as a perturbation and we can estimate the energies of the evolving states as the unperturbed energy splitting $\epsilon \sim \Delta$ plus a second order energy correction γ^2/Δ. The same result is obtained by considering the original hamiltonian with $2\Omega_0 = \sqrt{\Delta^2 + 4\gamma^2} = \Delta\sqrt{1 + 4\gamma^2/\Delta^2} \sim \Delta[1 + 2(\gamma^2/\Delta^2)] = \Delta + 2\gamma^2/\Delta$. The negativity oscillates with double frequency with respect to the state vectors, since its time evolution is dictated by s_0^2 and c_0^2. Therefore its approximated Rabi frequency will be $2\Omega_0 = \Delta + 2\gamma^2/\Delta = 3$ as it is clearly shown in fig.(3).

ENTANGLING POWER

The above qualitative remarks have been quantitatively proven by defining the entangling power [12] of the hamiltonian as:

$$E_P(T) = \sup_{t\in\tau} \mathcal{N}[\rho_T(t)] \qquad (19)$$

where τ is the time variation period of ρ, and has been chosen long enough to allow all the frequencies contributing the time evolution to be taken into account. The reference state ρ is a thermal state and the entangling power is computed as a function of the temperature. Fig.(5) shows its behaviour for a range of the ratio $1/\eta$ going from 0 to 10. Although the entangling power quickly decreases as temperature raises, it is however different from zero for a wide range of temperatures.
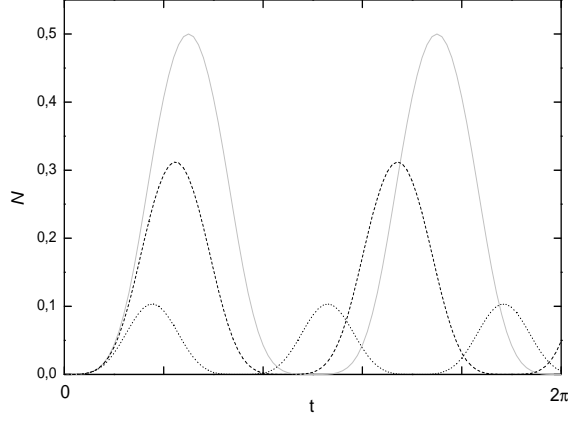
FIG. 3: Time variation of the negativity at $T = 0$ for different values of the detuning parameter: $\Delta = 0$ (thin gray line), $\Delta = 1$ (dashed line), $\Delta = 2$ (dotted line). Here the coupling constant is set as $\gamma = 1$.
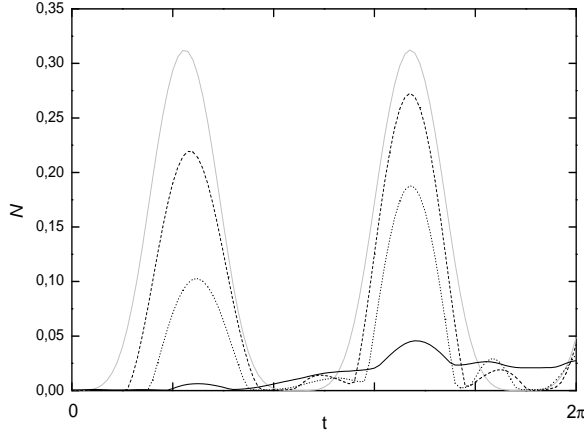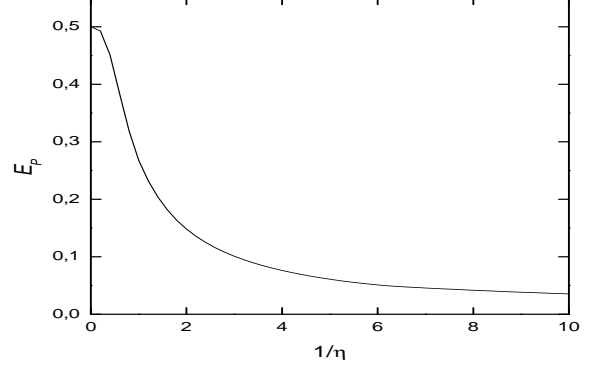

FIG. 5: The entangling power of the hamiltonian for the thermal state as a function of the temperature. As long as $1/\eta$ is lower than 1, E_P decreases very quickly, then it has a slow variation, and for high values of the parameter it remains almost constant.
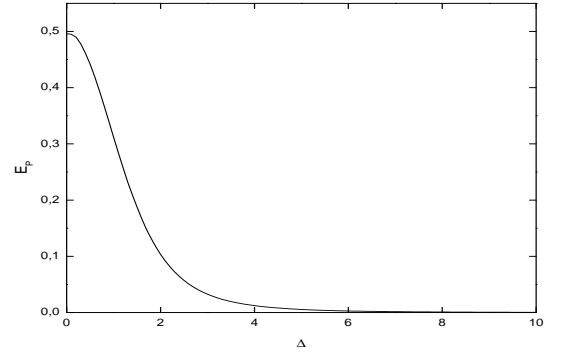

FIG. 4: Plot of the negativity time dependance in the case a fixed detuning $\Delta = 1$. Three different values of $1/\eta$ have been chosen: 0 (thin gray line), 0.5 (dashed line), 1 (dotted line), 5 (solid line). As in the previous case: $\gamma = 1$.

FIG. 6: Entangling power over all the off-resonant hamiltonian labeled by the detuning parameter Δ going from 0 to 10.

As a second step we studied the entangling power of a wider set of hamiltonian. As done in the previous section, we introduce the detuning parameter Δ in the hamiltonian and redefine the entangling power as:

$$E_P(\Delta) = \sup_{t \in \tau} \mathcal{N}\left[\rho^\Delta(t)\right] \tag{20}$$

where ρ is now the vacuum state, chosen as reference state. The result is displayed in fig.(6) in which we see the entangling power quickly decreasing down to zero, as Δ increases.

In order to combine the two previous results, we finally define the entangling power as:

$$E_P(T, \Delta) = \sup_{t \in \tau} \mathcal{N}\left[\rho_T^\Delta(t)\right] \tag{21}$$

The reference state is again the thermal state, and the 3-dimensional plot of E_P is represented in fig.(7). The most interesting feature of E_P is its monotonicity with respect to both T and Δ.

BUILDING A QUANTUM GATE

In this section we will examine the possibility of using the above analyzed entangling capabilities to build a quantum gate. The idea is to encode quantum information into the bosonic degrees of freedom by selecting a suitable finite-dimensional subspace of the Fock space. As mentioned in the introduction one would like to see whether with the proper encoding one can find out specific operating times such that the dynamics enacted by the Hamiltonian (1) amounts to a non-trivial transformation of the encoding subspace while acting trivially in the qubit factor. This latter requirement being of course due to the necessity of avoiding entanglement between
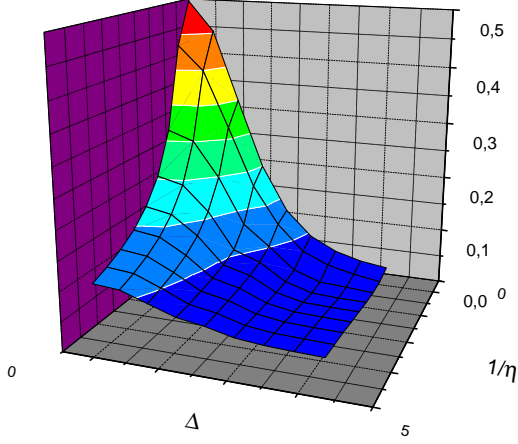
the bosons and the qubit that would in turn result into decoherence.

We consider as the encoded qubit the first two states of harmonic oscillator of the modes of the system studied so far. The logical computational basis is built by identifying: $|0\rangle_L = |g, 0\rangle_{12}$ and $|1\rangle_L = |g, 1\rangle_{12}$. This is in a sense the most natural basis choice. At variance with the previous calculations we prepare the system with the qubit in its ground state [18]. The two qubits evolution can be studied within the following scheme:

$$
\begin{aligned}
|00\rangle_L &= |g, 0, 0\rangle_{12} \\
|10\rangle_L &= |g, 1, 0\rangle_{12} \\
|01\rangle_L &= |g, 0, 1\rangle_{12} \\
|11\rangle_L &= |g, 1, 1\rangle_{12}
\end{aligned} \quad (22)
$$

FIG. 7: Three-dimensional plot of the entangling power for both Δ and $1/\eta$ going from 0 to 5. It is clear the monotone character of E_P.

The time dependence of the four basis states is given by:

$$
\begin{aligned}
|g, 0, 0\rangle_{12} &= |g, 0, 0\rangle_\pm \rightarrow |g, 0, 0\rangle_\pm \\
|g, 1, 0\rangle_{12} &= \frac{1}{\sqrt{2}}(|g, 1, 0\rangle_\pm + |g, 0, 1\rangle_\pm) \rightarrow \frac{1}{\sqrt{2}}(c_0(t)|g, 1, 0\rangle_\pm - is_0(t)|e, 0, 0\rangle)_\pm + \frac{1}{\sqrt{2}}|g, 0, 1\rangle_\pm \\
|g, 0, 1\rangle_{12} &= \frac{1}{\sqrt{2}}(|g, 1, 0\rangle_\pm - |g, 0, 1\rangle_\pm) \rightarrow \frac{1}{\sqrt{2}}(c_0(t)|g, 1, 0\rangle_\pm - is_0(t)|e, 0, 0\rangle_\pm) - \frac{1}{\sqrt{2}}|g, 0, 1\rangle_\pm \\
|g, 1, 1\rangle_{12} &= \frac{1}{\sqrt{2}}(|g, 2, 0\rangle_\pm - |g, 0, 2\rangle_\pm) \rightarrow \frac{1}{\sqrt{2}}(c_1(t)|g, 2, 0\rangle_\pm - is_1(t)|e, 1, 0\rangle_\pm) - \frac{1}{\sqrt{2}}|g, 0, 2\rangle_\pm
\end{aligned} \quad (23)
$$

With respect to the case in which the qubit was initially in its excited state, in this case the global state of the system evolves according to c_{n+-1} and s_{n+-1} instead of c_{n+} and s_{n+}.

Given this evolution table, it is possible to design a nontrivial unitary transformation on the computational basis by properly tuning the frequencies appearing int eq. (23). By inspecting at the dynamical evolution of the basis it is clear that one can enact the following gate:

$$
U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (24)
$$

Indeed, it is sufficient to find a time instant t_g for which $c_0(t_g) = -1, c_1(t_g) = 1$ and $s_0(t_g) = s_1(t_g) = 0$ to realize the above unitary transformation. These conditions can be achieved by choosing the variables t_g, γ, Δ such as the following equation is fulfilled:

$$
\frac{\Omega_0}{\Omega_1} = \frac{2N + 1}{2M} \quad (25)
$$

where N, M are natural numbers, that means that for Ω_0, Ω_1 we can choose $\forall q_0, q_1$ such that $q_0/q_1 \in \mathbb{Q}$. Thus once we

chose q_0, q_1, we get:

$$
\begin{aligned}
\gamma &= \frac{1}{2}\sqrt{q_1^2 - q_0^2} \\
\Delta &= \sqrt{2q_0^2 - q_1^2} \\
t_g &= \frac{\pi}{\frac{1}{2}q_0}(2N + 1) = \frac{\pi}{\frac{1}{2}q_1}2M
\end{aligned} \quad (26)
$$

whence we must add the conditions $q_1 > q_0$ and $2q_0^2 - q_1^2 \geq 0$ in order to have $\gamma, \Delta \in \mathbb{R}$. As an example consider $q_0 = \Omega_0 = 3$ and $q_1 = \Omega_1 = 4$; then $4\gamma^2 = 7, \Delta^2 = 2$ and $t_g = \pi + 2\pi K$.

The above sketched quantum gate however is the well known swap gate together with a single qubit π-phase shift. Unfortunately it is well known as well that this kind of gate, unless ancillary qubits are introduced, is not an entangling one. Anyway it is clear that no other unitary transformation is realizable within the proposed scheme. In fact as soon as $t \neq 0$ the evolution brings all the basis states out of the computational space, except for the first one. The fourth state comes back into the computational space only when $c_1 = 1$ and $s_1 = 0$ giving raise to the unchanged initial state. The

same goes for the the second and the third ($c_0 = 1$ and $s_1 = 0$), which have however the extra possibility of getting (-)exchanged ($c_0 = -1$ and $s_0 = 0$). The situation would not change even if we wanted to start with the qubit in its excited state. Again we would have to require that all $c_n = e^{\pm i\phi}$ and $s_n = 0$, where the minus sign is suitable only for the second and third state. Thus we could just have the same phase shift for all the states. This is due to the very fact that some of the coefficients (and so the frequencies) appear in every basis state, preventing the latter to dephase one from another. So one possibility to build a real quantum gate is to choose a different computational basis which, although it may not be scalable, allows however each state to evolve according to frequencies different from the others.

We can in principle build a phase-shifter gate by mapping the physical basis on the logical basis in a not straightforward way. The previous exercise and eq. (3) revealed that in order to have the basis last state's phase shifted alone, we need the evolution frequencies of this state to be different from the other states frequencies, at variance with what happens e.g. in eq. (23). A different choice of the basis would allows us to overcome this obstacle. One should choose the computational basis checking that: i) the states $|\psi\rangle_x = \frac{1}{N}(N_1|X, a_x, b_x\rangle_\pm + ... + N_n|X, y_x, z_x\rangle_\pm)$ have $a_x, b_x, ..., y_x, z_x \neq a_{x'}, b_{x'}, ..., y_{x'}, z_{x'}$, with $x, x' = 1, .., 4$ and $X = e, g$. ii) one is able to find t_g, γ, Δ such that $\Omega_{a_x}, ..., \Omega_{z_x}$ are commensurable frequencies. Unfortunately one has to deal with several commensurability conditions that are not obvious to be satisfied. Therefore we will not further consider this procedure any but we will focus on a second possibility, which is somehow more elegant.

We can choose a virtual bipartition [13] such that, with respect to this new bipartition, the swap gate becomes an entangling one. It is known indeed that the entanglement property of a state must be related to the choice of the subsystems between which the quantum correlation is measured [14, 21]. The swap operator itself can be an entangling one if we change the subsystems (the qubits in our case), by redefining the modes and the basis states. We know for example that the action of the swap operator on the Bell basis states induces a sign change of the antisymmetric state. Then we can change the bipartition in such a way that the new subsystems (qubits) are no more the two oscillation modes [21], but $|\chi\rangle$ and $|\lambda\rangle$ where $|\chi\rangle \otimes |\lambda\rangle = |\chi^\lambda\rangle = |\Phi(\Psi)^\pm\rangle = \frac{1}{\sqrt{2}}(|0, 0(1)\rangle \pm |1, 1(0)\rangle)$ with $\chi = \Phi, \Psi$ and $\lambda = +, -$. Thus by setting the computational basis as the Bell one and identifying:

$$|00\rangle_L = \frac{1}{\sqrt{2}}(|g, 0, 0\rangle_{12} + |g, 1, 1\rangle_{12})$$

$$|01\rangle_L = \frac{1}{\sqrt{2}}(|g, 0, 0\rangle_{12} - |g, 1, 1\rangle_{12})$$

$$|10\rangle_L = \frac{1}{\sqrt{2}}(|g, 0, 1\rangle_{12} + |g, 1, 0\rangle_{12})$$

$$|11\rangle_L = \frac{1}{\sqrt{2}}(|g, 0, 1\rangle_{12} - |g, 1, 0\rangle_{12}) \quad (27)$$

we can let them evolve according to eqs. (23) and (26), and get:

$$U' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (28)$$

Please note that, with respect to the previous case, now we are not simply changing the basis, we are indeed changing the computational subsystems. With this new choice the operator (28) is a controlled π-phase shift which can in principle be used for quantum computation tasks.

It is important to stress that the use of virtual bi-partitions makes now a non-trivial task the realization of single qubit gates i.e., those with a non-trivial action on just one of the (virtual) subsystems [21]. On the other hand the aim of this section was to discuss some issues related to gate building in the context of the dynamics (1) rather than proposing a universal set of gates. For this latter more ambitious goal, it is well known that some sort of non-linearity, possibly effective or even measurement-induced, between the bosonic modes is necessary [22].

CONCLUSIONS

In this paper we have studied the behaviour of the negativity of the quantum states of two iso-spectral bosonic modes symmetrically coupled with a common two-level system. By tracing over this latter we showed that the interaction creates maximally entangled states for $T = 0$ in short time ($\pi/2\gamma$). We then turned to study how entanglement production depends on the temperature by considering as initial bosonic state the thermal one. Not surprisingly the more the temperature raises the more entanglement decreases. Nevertheless, we observed how entanglement resists even up to relatively high T. A negativity peaks damping is also observed as the result of a progressive increase of the detuning of two-level system and the frequency of the bosonic modes: the more the system departs from the resonance condition i.e., detuning zero, the lower maximally achievable negativity gets.

The analysis of the entangling power of the interaction suggested us the way of designing a quantum gate adopting as computational basis the degrees of freedom of the two bosonic subsystems (the modes). The choice of the proper computational basis however turned up to be not straightforward, involving the introduction of virtual subsystems in order to be able to achieve a non-vanishing entangling power. We would like now to conclude by briefly commenting upon possible further investigations suggested by the study reported in this paper.

It is clear that one can consider the possibility of implementing the theoretical framework discussed in the above with systems different from the light-matter interaction usually chosen as the Jaynes-Cummings hamiltonian reference model. By properly adjusting the parameters Δ, γ one might

wonder whether, for example, is possible to set up a controllable electron-phonon interaction in a semiconductor heterostructure. Optical phonons can interact via a Frölich coupling with electrons in the valence band of a semiconductor superlattice designed to have only two sublevels. With respect to the present model the latter would involve more then one qubit i.e. atoms, of course. This would force one to study a generalization of the simple model discussed above. For instance one might have to consider some tailored periodic potential profile instead of a single two-level step.

The study of electron-mediated quantum correlations between phononic modes and their potential usage of quantum-information carriers is a particularly good illustration of the sort of logic which motivated this paper (see introduction). Indeed phonons represent an almost prototypical example of an incoherent decoherence-inducing system whereas electronic degrees of freedom are typically the ones proposed for the encoding the quantum information [15, 16]. Phonons, typically in a thermal state, are traced out while ways to enact coherent manipulations of electrons are contrived. In the future we would like to investigate whether an exchange of these roles of fermionic and bosonic degrees of freedom can be usefully figured out.

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